Network Mathematics - Why is it a Small World?

Oskar Sandberg

Networks

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- This is an abstraction which can be used to describe a lot of different systems (technical, physical, biological, sociological, etc. etc.).

Networks

Math	Graph	Vertex	Edge
CS	Network	Node	Link
Physics	System	Site	Bond
Sociology	Social Network	Actor	Tie
		Individual	Friendship
	WWW	Webpage	Link (d)
	Internet	Site	Connection
		Network	Bridge
	Road System	Crossing	Road

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- Randomly evolved: The Web, social networks.
- Somewhere in between: The Internet, P2P Networks.

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- When designing structured networks, questions are usually algorithmic. (How do I create a network with this property?)
- When studying randomly generated networks questions tend to analytic. (Does the network have this property?)

Random Graph Theory

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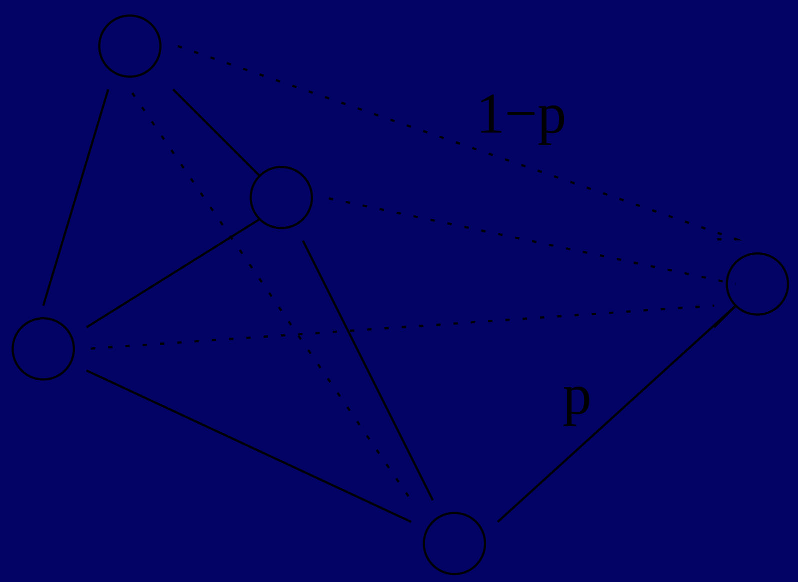
The simplest model for a random graph $G(n,p) = (V,E)$:

- $V = \{0,1,2,\ldots,n\}$
- $u \leftrightarrow v$ (that is $(u,v) \in E$) independently and with probability p for every pair of vertices u and v.



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- If $p > 1/n$ “most” of the vertices form one connected cluster.
- If $p > \log n/n$ all of the vertices are connected.
- The “diameter” of the connected cluster is $\log n$.

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In recent years, new models have been introduced for networks with various properties.

Example: Preferential Attachment

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- Vertices join the graph one by one, each connecting to those already in the network.
- The new node chooses who to connect to with a probability proportional to each older vertices current degree.

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- The "Small World Phenomenon" is that many naturally occurring networks have a small graph diameter.
- It was famously illustrated for social networks by Stanley Milgram in 1967.
- He experimented by having volunteers in Omaha, Nebraska forward letters to a stockbroker in Boston through friends.
- Milgram reported that on average the packages reached their destination in only six steps.

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- As noted, however, they are not a good model for social networks.
- It isn't possible to *search* in them.

Kleinberg's Model

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- α tunes the degree of “locality” the shortcuts.
- Route using *greedy routing*: step to the neighbor which is closest to destination.

Kleinberg's Model, cont.

Efficient routing is possible when α is such that:

$$\mathbf{P}(x \rightsquigarrow w) \propto \frac{1}{\text{\# nodes closer to } x \text{ than } w}$$

This can be seen to be $\alpha = d$, where d is the dimension of the space (2 in the simulations).

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- Kleinberg's result is mostly negative: for the vast majority of networks, searching is not possible.
- Why should one expect real-world networks to have the necessary edge distribution?

Some Math

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- Consider only the relative size of the first k numbers drawn.
- These have a random order: each is equally likely to be the biggest of them.
- Thus the k-th number has probability $1/k$ of being the biggest one yet.

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- Let u associate with each other node v a random quantity representing u's interest in v.
- Let $u \leftrightarrow v$ if u is more interesting to v than any node which is closer.

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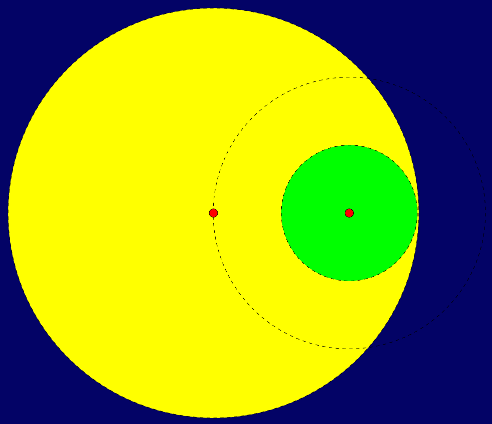
  This is now independent for each v.

- Expected number of shortcuts from each node is $\log n$.
- One can see that greedy routing takes $O(\log n)$ steps on a graph generated like this.

A Proof



- If d is the distance between u and v, the yellow disk is the vertices within $(3/2)d$ of u and the green within $d/2$ of v.

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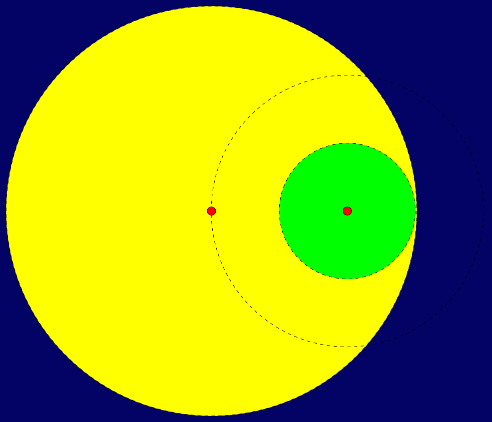

- If d is the distance between u and v, the yellow disk is the vertices within $(3/2)d$ of u and the green within $d/2$ of v.
- u must have a shortcut to the very “most interesting” vertex in the yellow disk.

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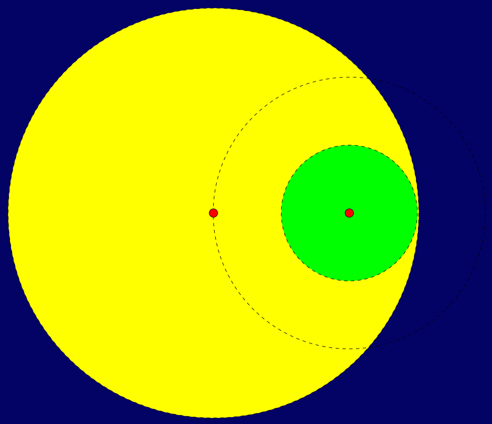

- If d is the distance between u and v, the yellow disk is the vertices within $(3/2)d$ of u and the green within $d/2$ of v.
- u must have a shortcut to the very "most interesting" vertex in the yellow disk.
- The probability that that vertex is in the green part is $1/9$.

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- With each vertex u we associate a position $p(u)$ in some "space of interests".
- Let u's interest in v be the inverse of $|p(u) - p(v)|$.
- That is: $u \leftrightarrow v$ if $p(u)$ is closer to $p(v)$ than p of any node closer to u to than v.

The Double Clustering Graph

Definition 1 *Let* $(x_i)_{i=1}^{n}$ *and* $(y_i)_{i=1}^{n}$ *be two sequences of points without repetition in possibly different spaces* M_1 *and* M_2 *with distance functions* d_1 *and* d_2 *respectively. The digraph* $G = (V, E)$ *is constructed as follows:*

- $V = \{1, 2, \ldots, n\}$.
- $(i, j) \in E$ *if for all* $k \in V$, $k \neq i, j$:

$$d_1(x_i, x_k) < d_1(x_i, x_j) \Rightarrow d_2(y_i, y_k) \geq d_2(y_i, y_j)$$

(Make undirected by removing directionality of the edges.)

Conclusion

- Simple probabilistic models can explain complicated network structures.
- Finding such models can help with both network analysis and design.
- It involves a lot of interesting mathematics.

Conclusion

The end